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Brane Cosmology, Varying Speed of Light and Inflation in Models With One or More Extra Dimensions[1]

D. A. Steer[2,4] and M. F. Parry[3]

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We summarize the approach to brane cosmology known as "mirage cosmology" and use it to determine the Friedmann equation on a 3-brane embedded in different bulk spacetimes all with one or more extra dimensions. Usually, when there is more than one extra dimension the junction conditions, central to the usual brane world scenarios, are difficult to apply. This problem does not arise in mirage cosmology because the brane is treated as a "test particle" in the background spacetime. We discuss in detail the dynamics of a brane embedded in two specific 10D bulk spacetimes, namely Sch-$AdS_5 \times S_5$ and a rotating black hole, and from the dynamics—which are now rather more complicated since the brane can move in all the extra dimensions—determine the new "dark fluid" terms in the brane Friedmann equation. Some of these, such as the cosmological constant term, are seen to be bulk dependent. We then show explicitly how this mirage cosmology approach matches with the familiar junction condition approach when there is just one extra dimension. The issue of a varying speed of light in mirage cosmology is addressed and we find a scenario in which $c_{\rm eff}$ always increases, tending asymptotically to c_0 as the universe expands. Finally some comments are made regarding brane inflation and limitations of the mirage cosmology approach are also discussed.

KEY WORDS: brane cosmology; dimensions; bulk 5D metric.

1. INTRODUCTION

Recently there has been much interest in the idea that our universe may be a 3-brane embedded a spacetime of five or more dimensions. In particular, following the work of Randall and Sundrum (1999a,b), brane cosmology in models with one infinite extra dimension has been studied in depth (Binétruy *et al.*, 2000a,b; Cline *et al.*, 1999; Csaki *et al.*, 1999; Flannagan, *et al.*, 2000; Ida, 2000; Kraus, 1999).

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[2] Laboratoire de Physique Théorique, Université Paris XI, Orsay Cedex, France.

[3] Theoretische Physik, Ludwig-Maximilians Universität, Theresienstraße 37, München, Germany.

[4] To whom correspondence should be addressed at Laboratoire de Physique Théorique Bat 210, Université Paris XI, F91405, Orsay Cedex, France; e-mail: steer@th.u-psud.fr.

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There are essentially two distinct approaches to determining this brane cosmology. In the first (e.g. (Binétruy *et al.*, 2000a,b)), coordinates are chosen relative to the brane which is thus at a fixed position in the extra dimension. The bulk 5D metric, on the other hand, is time dependent and this time dependence induces a time dependence on the brane via the junction conditions. The resulting Friedmann equation on the brane is found to have a characteristic ρ^2 term (Binétruy *et al.*, 2000a,b), where ρ is the energy density in matter which is assumed to be confined on the brane, as well as a "dark" radiation term originating from the Weyl tensor in the bulk (Shiromizu, 2000).

In the alternative but equivalent approach (e.g. (Ida, 2000; Kraus, 1999)), the bulk is static and the brane dynamical: the brane moves through a time-independent bulk metric. If the vacuum Einstein equations hold in the bulk, and if one imposes that our universe brane has the symmetry of a 3-sphere, then it is possible to prove that the bulk must be Sch-AdS$_5$ (Bowcock *et al.*, 2000; Carter and Uzan, 2001). Thus the brane divides two regions of Sch-AdS$_5$ and its dynamics can be determined from the junction conditions. For reasons which will be summarized in section 2, this motion of the brane through the bulk induces cosmology on the brane even if *no* matter confined to the brane. This is sometimes called the "mirage" effect (Kehagias and Kiritsis, 1999) because cosmological evolution is not necessarily sourced by the local energy density of the brane. When matter is also included on the brane, the resulting Friedmann equation which one obtains with this approach is identical to that obtained when the brane is static and the bulk time dependent. (The explicit coordinate transformation linking the two approaches may be found in Mukohyama, *et al.* (2000).) The dark radiation term can now be understood as being due to the motion of the brane.

Typically in both these approaches, it is assumed that the brane divides the bulk into two identical pieces—that is, there is Z_2 symmetry across the brane. This assumption can easily be relaxed and in particular Z_2 symmetry will be broken if the brane is charged and couples to a 4-form field living in the bulk (Carter and Uzan, 2001). In context of the moving brane approach, one would therefore have different cosmological constants $\Lambda_\pm$ and masses $M_\pm$ parametrizing the Sch-AdS$_5$ spacetimes on each side of the brane, and thus the brane dynamics would be altered. In particular, it is possible to show (Carter and Uzan, 2001; Ida, 2000; Kraus, 1999) that the resulting Friedman equation now has an extra dark radiation term with energy density proportional to $[\Lambda]/a^4$ (where $[\Lambda] = \Lambda_+ - \Lambda_-$) as well as a new dark fluid term with energy density proportional to $[M][\Lambda]/a^8$ (see also section 4.2).

One of the questions we try to address here is the following: if the brane is embedded in a spacetime of more than five dimensions, what dark fluid terms are generated in the brane Friedmann equation? To answer this question we work in the frame in which the brane is dynamical, moving through a static or stationary bulk. In the usual brane world scenarios, it is important to satisfy the brane junction conditions. This reflects that the background metric must be consistent with the presence of the brane. Unfortunately, it is often not straightforward to apply the junction conditions when there is more than one extra dimension since the results typically depend on the thickness of the defect, ϵ, and are not well defined as $\epsilon \rightarrow 0$. However, this is not fatal to our program. For objects with codimension greater than one, it becomes reasonable to treat them as "test particles" in the background spacetime. In other words, there is no back-reaction to solve for. This is analogous to the case of planetary orbits where the Earth, for instance, is treated as a point particle moving in the spacetime metric generated by the sun (see section 4.1).

Our particular approach to brane cosmology is to consider D3-branes in type IIB string theory. Such D3-branes are attractive because they are stable and, by construction, matter is localized on them. Furthermore, an action can, within certain approximations, be derived (Bachas, 1998); it consists of the Dirac-Born-Infeld (DBI) action plus a Wess-Zumino term. (For slowly moving branes, the D-brane action has been used extensively to study the properties of near extremal black holes (Maldacena, 1996)). The only caveat is that D3-branes are BPS states so that one must eventually provide a prescription for supersymmetry-breaking. As for the background in which the branes move, this is consistently determined from the low-energy string action. Here it is a 10D supergravity action and we consider a Sch-AdS$_5 \times$S$_5$ bulk metric and a rotating black hole solution, both of which can be thought of as being generated by a stack of D3-branes. The approach we describe was coined "mirage cosmology" (MC) and developed in depth by Kehagias and Kiritsis (Kehagias and Kiritsis, 1999; Kiritsis, 1999a,b) and extended by others (Papantonopoulos and Pappa, 2002; Youm, 2000,2001; Brax and Steer 2001).

One of the purposes of this paper is to try to introduce MC to cosmologists who are perhaps more familiar with 5D brane cosmology. (As such, a part of the work presented here will follow Kehagias and Kiritsis (1999)). The first important point is that the MC approach is "passive." As intimated above, the D3-brane is assumed *not* to back-react on the bulk. In this sense, this approach is very similar to that used to determine the dynamics of cosmic topological defects. It differs from the "active" 5D case where the junction conditions include the back-reaction of the brane on the bulk. We dub this approach the junction conditions (JC) approach. Secondly, notice that when there is more than one extra dimension, the brane has much more freedom in its motion. For example, in Sch-AdS$_5 \times$S$_5$ the brane may not only move along the radial coordinate but also around the S_5. However, it turns out that the brane angular momentum ℓ is conserved around this S_5 (section 4.1). In section 4.2, we set $\ell = 0$ and discuss how the Friedmann equation obtained from this MC approach is linked to that obtained via the junction conditions. In order to make this link though, it is necessary to consider the situation in which Z_2 symmetry is broken (Carter and Uzan, 2001; Kraus, 1999) since D-branes are charged under Ramond-Ramond fields living in the bulk (Bachas, 1998).

A final purpose of this paper is to try to present new results on mirage cosmology. In particular, in section 5, we consider mirage cosmology in a rotating black hole background and comment on other work in this area. The possibility of a varying speed of light is discussed in section 6.1. In section 6.2 we consider brane inflation when the bulk is generated by a “brane gas” and make other comments regarding inflation in mirage cosmology. Finally conclusions are given in section 7 where we discuss some of the limitations of this approach to brane cosmology, perhaps most importantly, the lack of brane self-gravity (see, however, Brax and Steer, 2002).

2. EFFECTIVE COSMOLOGY FROM BRANE MOTION

We begin by introducing our notation and explaining briefly the “mirage” effect. Consider an infinitely thin p-brane in a $(D+1)$-dimensional spacetime. The following index convention will be used to label objects:

$$\underbrace{0\,\underbrace{\overbrace{1\cdots p}^{a}}_{i}\;\overbrace{p+1\cdots D}^{A}}_{\mu} \tag{2.1}$$

The $D+1$ spacetime coordinates are denoted by x^μ with $x^0 \equiv t$ being the time coordinate, and the background metric $g_{\mu\nu}(x)$ has signature $(-++\cdots+)$. As the brane moves it sweeps out a $p+1$ dimensional world-sheet labeled by coordinates σ^i. The position of the brane in the background spacetime is $x^\mu = X^\mu(\sigma)$ so that the induced metric on the brane is

$$\gamma_{ij}(\sigma) = g_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^i}\frac{\partial X^\nu}{\partial\sigma^j}. \tag{2.2}$$

We consider an infinitely long straight brane parallel to the x^i-hyperplane, but free to move along the perpendicular coordinates x^A. Hence a natural choice of intrinsic coordinates[5] is $\sigma^i = x^i$, and the brane motion is described by

$$X^i = x^i, \qquad X^A = X^A(t). \tag{2.3}$$

This is known as the static gauge. If one wanted to study perturbed branes, the relevant embedding would be $X^i = x^i$, $X^A = X^A(x^i)$ (see (Boehm and Steer, 2002)).

Whether or not the induced brane metric γ_{ij} is spatially homogeneous and isotropic depends on the background metric. The background line-elements

[5] Of course any brane action must be invariant under reparametrizations $\sigma^i \to \tilde{\sigma}^i$, hence there is freedom to choose the $p+1$ coordinates so as to simplify the resulting equations of motion as much as possible.

considered here are either static or stationary and take the form

$$ds^2 = g_{00}\,dt^2 + \sum_a g_{aa}(dx^a)^2 + 2g_{0,p+1}\,dt\,dx^{p+1} + \sum_A g_{AA}(dx^A)^2 \tag{2.4}$$

with

$$g_{\mu\nu} = g_{\mu\nu}(x^A). \tag{2.5}$$

The induced metric γ_{ij} is then

$$\begin{aligned}\gamma_{00} &= g_{00} + 2g_{0,p+1}\dot{X}^{p+1} + \sum_A g_{AA}\dot{X}^A\dot{X}^A\\ \gamma_{0a} &= 0\\ \gamma_{ab} &= g_{aa}\delta_{ab} \qquad \text{(no sum)}\end{aligned} \tag{2.6}$$

where $\cdot = \partial/\partial t$, and the metric coefficients are evaluated on the brane, i.e. $g_{\mu\nu}(x^A) \to g_{\mu\nu}(X^A(t))$. It follows that $\gamma_{ij} = \gamma_{ij}(t)$ and, in particular, that the brane is spatially flat. To consider a curved brane (see Youm, 2001; Brax and Steer 2002a).

Now let $p = 3$. We consider backgrounds for which $g_{11} = g_{22} = g_{33} \equiv g_d$ so that the brane metric γ_{ij} is indeed spatially homogeneous and isotropic.[6] The induced line-element on the brane is

$$\begin{aligned}ds^2 &= \gamma_{ij}dx^i dx^j\\ &= \gamma_{00}(t)\,dt^2 + g_d(X^A(t))\,d\mathbf{x}^2\\ &\equiv -d\tau^2 + a^2(\tau)\,d\mathbf{x}^2,\end{aligned} \tag{2.7}$$

where the brane time τ is defined by

$$d\tau = \sqrt{-\gamma_{00}(t)}\,dt \tag{2.8}$$

and the brane scale factor $a(\tau)$ by

$$a^2(\tau) = g_d(X^A(t(\tau))). \tag{2.9}$$

This is the "mirage" effect: the brane motion has generated a scale factor $a(\tau)$ in the brane independently of whether or not there is matter on the brane. The details of $a(\tau)$ depend both on the background, through g_d, and on the brane motion, through $X^A(t(\tau))$. Finally the Friedmann equation is given by

$$H^2 = \left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = \frac{1}{4}\frac{1}{|\gamma_{00}|}\frac{1}{g_d^2}\left[\sum_A\left(\frac{\partial g_d}{\partial X^A}\dot{X}^A\right)\right]^2 \equiv \frac{8\pi G_4}{3}\rho_{\text{eff}} \tag{2.10}$$

which defines an effective energy density.

[6] If $g_{11} \neq g_{22} \neq g_{33}$, then the brane metric is homogeneous but anisotropic. Such a situation occurs when there is a nonzero bulk magnetic NS field (Youm, 2000). Of cosmological interest, would be situations in which the motion of the brane (i.e. its expansion) leads to isotropization of the brane.

3. BRANE ACTION AND THE BACKGROUND METRIC

We assume that our universe is a D3-brane in type IIB string theory, and that the background spacetime in which it moves is generated by all the allowed degrees of freedom. In the low-energy limit we have a 10D supergravity action from which the bulk metric may be determined (Stelle, 1997). Apart from section 6.2, the bulk will be assumed to contain a charged stack of many coincident D3-branes which generate, amongst other possibilities, a Sch-AdS$_5\times$S$_5$ bulk metric (Kiritsis, 1996).

The "universe-brane" itself (on which, by construction, gauge fields are confined) can also couple to many different objects and determining its action is still an active area of research. However, in the simplest case we can think of our universe as a probe D3-brane whose action is given by (Bachas, 1998)

$$S = S_{\mathrm{DBI}} + S_{\mathrm{WZ}} = -\lambda \int d^4\sigma \sqrt{-\det(\gamma_{ij} + (2\pi\alpha')F_{ij} - B_{ij})} - e \int \mathcal{C}_4, \quad (3.1)$$

where λ is the brane tension, α' is the string tension and e is the brane charge density. The dilaton has not been included because it is constant in the supergravity solutions being considered. The "kinetic" term, the DBI action, is the volume of the brane trajectory (the Nambu-Goto piece) modified by the presence of the pull-back of the Neveu-Schwarz antisymmetric two-form B_{ij}, and worldvolume antisymmetric gauge fields F_{ij}. (The latter arise due to open strings which may connect the probe and stack D3-branes.) Thus, for example, if there is radiation on the brane, $F_{ij} \neq 0$, and the brane dynamics will be altered relative to the case of a brane with no radiation.[7] The modified dynamics will in turn change the Friedmann equation (as explained in section 2) which will thus contain terms reflecting the presence of the radiation 2 (Kehagias and Kiritsis, 1999 and Youm, 2001) (see also section 4.4).

Note, however, that the brane action as it stands does not allow for arbitrary matter content. Thus, as presented so far, MC cannot provide a full account of the evolution of our universe. (It would, after all, be unbelievable if our current cosmology, based upon 4D gravity and the local matter density, could be emulated solely by the motion of a brane in a higher-dimensional background.) However, MC may well have a role to play where our understanding is not well established, namely at early and late (future) times. Additionally, (3.1) provides a springboard for more phenomenological approaches. We return to these themes in (6.2).

[7] This is exactly the same effect as in the case of current carrying cosmic strings. With no current, the string action is the NG action. With a current, the action is changed and it may lead to very different cosmic string dynamics—for example, stable loops called vortons may now be formed (Davis *et al.*, 2000).

The Wess-Zumino term in (3.1) is required since the probe D3-brane is charged under Ramond-Ramond gauge fields living in the bulk. Here it takes the simple form given in (3.1) because we assume that the stack is the sole source of RR fields. Thus the only contribution is from a 4-form $\mathcal{C}_4$ and

$$S_{\mathrm{WZ}} = -e \int \mathcal{C}_4 = -e \int \frac{1}{4!} C_{\mu\nu\rho\tau} \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} \frac{\partial X^\rho}{\partial \sigma^k} \frac{\partial X^\tau}{\partial \sigma^\ell}\, d\sigma^i\, d\sigma^j\, d\sigma^k\, d\sigma^\ell, \quad (3.2)$$

where the gauge field $C_{\mu\nu\rho\tau}$ is obtained from the corresponding field strength $F_{\alpha\mu\nu\rho\tau}$ (which off the brane and for $r > 0$ is a solution of $\nabla^\alpha F_{\alpha\mu\nu\rho\tau} = 0$). By virtue of the coordinate choice (3.1), we have assumed that the probe brane is parallel or antiparallel to the stack. Supersymmetry of the total system remains unbroken only in the parallel case when the brane is BPS implying that $e = \lambda$ (Bachas, 1998). The antiparallel case corresponds to the probe being an antibrane and to $e = -\lambda$. However, in order to make comparisons with more phenomenological brane world scenarios and ones using the JCs, we will write more generally

$$e = q\lambda. \quad (3.3)$$

4. MIRAGE COSMOLOGY IN SCHWARZSCHILD-ADS$_5 \times$ S$_5$

A particularly illustrative background in which to apply the mirage cosmology approach is Sch-AdS$_5 \times$S$_5$, since if one dropped the S_5 piece it would correspond to the background metric used in the moving brane approach to 5D brane cosmology described in the introduction (Kraus, 1999). Thus the dynamics of the brane around the S_5 should give an indication of which dark fluid terms are generated in models with more than one extra dimension. The Sch-AdS$_5 \times$S$_5$ metric is given by

$$ds^2 = \frac{r^2}{L^2}\left(- f(r)\, dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{L^2 dr^2}{f(r) r^2} + L^2\, d\Omega_5^2 \quad (4.1)$$

where

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^4 \quad (4.2)$$

and $d\Omega_5^2 = h_{IJ}(\phi)\phi^I \phi^J$ is line element of the unit 5-sphere described by coordinates ϕ^I, $I = 0 \ldots 5$. The metric satisfies the 10D Einstein equations with $\Lambda \equiv -16/L^2$ and $r_0^4 \equiv 2GML^2$ gives the black hole mass M, with G the 10D Newton constant. In this background the radial position of the brane determines the brane scale factor a since from (2.9)

$$a = \frac{r}{L}. \quad (4.3)$$

We take $r_0/L \leq a \leq \infty$, though it would be interesting to determine what happens for a brane that crosses the black hole horizon.

In order to obtain $a(\tau)$ the brane dynamics $r(\tau)$ must be calculated. Initially (section 4.1) we assume that there is no radiation on the brane ($F_{ij} = 0$) and turn off the bulk NS fields ($B_{ij} = 0$). Then the Friedmann equation which follows from (4.3) will contain only dark fluid terms. Some of these terms (sections 4.1 and 4.2) will be seen to be the familiar dark fluid terms of 5D brane worlds mentioned in the introduction. However, other terms arise from the nontrivial dynamics of the brane around the S_5 and they can lead to some interesting effects (section 4.3). In section 4.2 we define precisely the link between this MC approach to brane world cosmology and the junction condition approach used in 5D. Finally in section 4.4 we consider briefly the case of nonzero F_{ij}. Parts of sections 4.1 and 4.4 follow closely Kehagias and Kiritsis (1999).

4.1. Brane Dynamics With No Matter

To maintain some generality we write the bulk metric line element as

$$ds^2 = g_{00}(r)\,dt^2 + g_d(r)\,d\mathbf{x}^2 + g_{rr}(r)\,dr^2 + g_s(r)\,d\Omega_5^2 \tag{4.4}$$

which includes the Sch-AdS$_5\times$S$_5$ metric of (4.1). The only nonzero component[8] of $C_{\mu\nu\rho\tau}$ is then $C_{0123}(r) \equiv C_4(r)$. Thus in the gauge (3.1), $\mathcal{C}_4 = C_4(r)d^4x$ and, with $F_{ij} = B_{ij} = 0$, the action (3.1) defines a dimensionless Lagrangian $\mathcal{L}$ through

$$S = -\lambda \int d^4x\sqrt{-\det(\gamma_{ij})} - q\lambda \int d^4x\; C_4 \equiv \lambda V_3 \int dt\mathcal{L}, \tag{4.5}$$

where $V_3 = \int d^3x$. Using (2.2) gives

$$\mathcal{L} = -\sqrt{-g_d^3\gamma_{00}} - qC_4 \equiv -\sqrt{\mathcal{A} + \mathcal{B}\dot{r}^2 + \mathcal{C}h_{IJ}\dot{\phi}^I\dot{\phi}^J} + \mathcal{E} \tag{4.6}$$

with

$$\mathcal{A} = -g_d^3 g_{00}, \quad \mathcal{B} = -g_d^3 g_{rr}, \quad \mathcal{C} = -g_d^3 g_s, \quad \mathcal{E} = -qC_4. \tag{4.7}$$

By inspection, since $\mathcal{L}$ is not explicitly time dependent and the ϕ-dependence is confined to the kinetic term for $\dot{\phi}$, the brane geodesics are parametrized by a conserved energy E and an angular momentum ℓ^2 given, respectively, by

$$E = \frac{\partial\mathcal{L}}{\partial\dot{r}}\dot{r} + \frac{\partial\mathcal{L}}{\partial\dot{\phi}^I}\dot{\phi}^I - \mathcal{L}, \qquad \ell^2 = h^{IJ}\frac{\partial\mathcal{L}}{\partial\dot{\phi}^I}\frac{\partial\mathcal{L}}{\partial\dot{\phi}^J}. \tag{4.8}$$

[8] Actually, the self-duality condition for the field strength for $p = 3$ means the 4-form will also have nonzero components in the S_5-directions, but these do not contribute to the WZ term in the static gauge.

Solving these expressions for $\dot{\phi}$ and $\dot{r}$ gives

$$h_{IJ}\dot{\phi}^I\dot{\phi}^J = \frac{\mathcal{A}^2\ell^2}{\mathcal{C}^2(E+\mathcal{E})^2}, \qquad \dot{r}^2 = -\frac{\mathcal{A}}{\mathcal{B}}\left[1+\frac{\mathcal{A}}{\mathcal{C}}\frac{(\ell^2-\mathcal{C})}{(E+\mathcal{E})^2}\right]. \tag{4.9}$$

The brane time τ is then obtained on substitution of (4.9) into (2.8):

$$d\tau^2 = \frac{1}{g_d^3}(\mathcal{A}+\mathcal{B}\dot{r}^2+\mathcal{C}h_{IJ}\dot{\phi}^I\dot{\phi}^J)dt^2 = \frac{\mathcal{A}^2}{(E+\mathcal{E})^2}\frac{1}{g_d^3}dt^2 \tag{4.10}$$

and the Friedmann equation (2.10) becomes

$$H^2 = -\frac{g_d(g_d')^2}{4\mathcal{A}\mathcal{B}\mathcal{C}}\left[\mathcal{A}(\ell^2-\mathcal{C})+\mathcal{C}(E+\mathcal{E})^2\right]. \tag{4.11}$$

The specific forms of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$ and a for the Sch-AdS$_5\times$S$_5$ metric (4.1) are

$$\mathcal{A} = \frac{r^8}{L^8}f = a^8\left(1-\frac{X^4}{a^4}\right), \qquad \mathcal{B} = -\frac{r^4}{L^4 f} = -a^4\left(1-\frac{X^4}{a^4}\right)^{-1},$$
$$\mathcal{C} = -\frac{r^6}{L^4} = -a^6L^2, \qquad \mathcal{E} = q\left(\left(\frac{r}{L}\right)^4-\frac{X^4}{2}\right) = q\left(a^4-\frac{X^4}{2}\right) \tag{4.12}$$

where $X = r_0/L$ and we have used the expression for the 4-form in Sch-AdS$_5\times$S$_5$ (Kehagias and Kiritsis, 1999)

$$C_4(r) = -\frac{r^4}{L^4}+\frac{r_0^4}{2L^4} \tag{4.13}$$

Finally, the Friedmann (4.11) equation is

$$H^2 = \frac{q^2-1}{L^2}+\frac{X^4L^2}{a^4}+L^6\left(\frac{\tilde{E}}{a^4}\right)\left(\frac{\tilde{E}}{a^4}+\frac{2q}{L^4}\right)+\ell^2\frac{L^4}{a^6}\left(\frac{X^4L^2}{a^4}-\frac{1}{L^2}\right), \tag{4.14}$$

where we now rescaled the scale factor a by a factor of L to give it dimensions of length and the constant part of $\mathcal{E}$, essentially electrostatic energy, has been absorbed into the energy so that $\tilde{E} = E - qX^4/2$. Note that the first term is the effective cosmological constant on the brane but that this vanishes when $q = \pm 1$.

The dependence of this Friedmann equation on ℓ will be discussed in subsection 4.3 where we will comment on the final term of (4.14) which contributes a negative energy density for $r > r_0$. We now focus on the case $\ell = 0$.

4.2. $\ell = 0$: "Mirage" Versus Brane World Cosmology

When $\ell = 0$ the D3-brane has no dynamics about the S_5 and hence its motion is effectively constrained to a Sch-AdS$_5$ bulk with metric

$$ds_5^2 = \frac{r^2}{L^2}\left(-f(r)\,dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{L^2 dr^2}{f(r) r^2}. \tag{4.15}$$

It is straightforward to work out the Friedmann equation resulting from the MC approach in this case: it is given in (4.14) where one must set $\ell = 0$. (Note that the cosmological constant corresponding to (4.15) is now $\Lambda = -6/L^2$ and that $r_0^4 \equiv 2G_5 ML^2$ where G_5 is the 5D Newton constant. In obtaining the Friedmann equation we use (4.13) which also holds in 5D.) However, as was discussed in the introduction, the brane dynamics—including the back-reaction of the brane on the bulk—are also known in this case (i.e. with bulk metric (4.15)) from the JC approach (Carter and Uzan, 2001; Ida, 2000; Kraus, 1999). The purpose of this section is to compare these two Friedmann equations. Are they related in any way? And if so, how do the different parameters relate to one another? In other words, when there is only one extra dimension, how does the "test brane" MC approach compare with the "exact" JC approach?

Before making this comparison, note three important points. Firstly, since the D3-branes we are considering in this paper are charged and couple (minimally) to a 4-form field living in the bulk, one must consider, in the JC approach, a setup in which Z_2 symmetry is broken—see comments in the introduction and Carter and Uzan (2001). Thus the brane, with charge e_4 say, divides two different regions of Sch-AdS$_5$ with cosmological constants and masses $(\Lambda_\pm, M_\pm)$. Secondly, in deriving (4.14) we have set $F_{ij} = B_{ij} = 0$ and so there is no matter on the brane, thus we must set $\rho = p = 0$ in the JC approach. Finally, we have considered a flat brane, so $k = 0$ in the JC approach.

Once these conditions are imposed, the resulting Friedmann equation calculated in Carter and Uzan (2001) using the junction conditions depends on G_5 and e_4 as well as five other dimensionful quantities: the brane tension λ—which is the same as that in (3.1)—and $\Lambda_\pm$, $M_\pm$. In fact the combinations which appear are $\langle M\rangle$, $[M]$, $\langle\Lambda\rangle$ and $[\Lambda]$ where $\langle x\rangle \equiv (x_+ + x_-)/2$ and $[x] \equiv x_+ - x_-$. A final important identity relates the force on the brane $e_4\langle F\rangle$ to the jump in cosmological constant (Carter and Uzan, 2001):

$$[\Lambda] = 6\pi^2 e_4 G_5 \langle F\rangle. \tag{4.16}$$

Here F is defined through the physical 5-form field strength $F_{\mu\nu\rho\sigma\tau} = F\epsilon_{\mu\nu\rho\sigma\tau}$ corresponding to the bulk gauge field $A_{\mu\nu\rho\sigma}$. From the 5D SUGRA equations of motion it is straightforward to show that $F = K_1 r^3/L^4$, where the constant K_1 is dimensionless, and hence that A_{0123} is, up to a multiplicative constant, just C_{0123} of (4.13) used in the MC approach (see (Boehm and Steer, 2002)). Furthermore, from the equations of motion, the effective cosmological constants are easily seen to be $\Lambda_\pm = \Lambda \pm K_2 F_\pm^2$ where K_2 is another numerical constant and $\Lambda = -6/L^2$. Notice from (4.16) that if the brane is uncharged, $e_4 = 0$, then $[\Lambda] = 0$ so that there is no force on the brane. Finally the resulting Friedmann equation is (Carter and Uzan, 2001)

$$H^2 = \frac{\Lambda_4}{3} + \frac{2G_5\langle M\rangle}{a^4} + \left(\frac{3}{8\pi\lambda}\right)^2 \frac{[M]}{a^4}\left(\frac{[M]}{a^4} + \pi^2 e_4\langle F\rangle\right), \tag{4.17}$$

where a is the dimensionful scale factor, and the effective cosmological constant Λ_4 is given by[9]

$$\frac{\Lambda_4}{3} = \frac{\langle\Lambda\rangle}{6} + \frac{1}{4}\left(\frac{8\pi\lambda}{3}\right)^2 G_5^2 + \frac{1}{4}\left(\frac{3}{8\pi\lambda}\right)^2 \left(\pi^2 e_4\langle F\rangle\right)^2. \tag{4.18}$$

If Z_2 symmetry is imposed, i.e. $[M] = [\Lambda] = 0$, then (apart from the cosmological constant term) the Friedmann equation (4.17) contains only the familiar dark radiation term coming from the electric part of the Weyl tensor. With no Z_2 symmetry there is the extra dark radiation term plus a contribution $\sim a^{-8}$, as mentioned in the introduction.

How does this Friedmann equation (4.17) compare with the Friedmann equation (4.14) obtained from the DBI action when $\ell = 0$? That equation is

$$H^2 = \frac{q^2-1}{L^2} + \frac{2G_5 M}{a^4} + L^6\left(\frac{\tilde{E}}{a^4}\right)\left(\frac{\tilde{E}}{a^4} + \frac{2q}{L^4}\right). \tag{4.19}$$

Notice first that (4.19) and (4.17) have a very similar form, and in particular exactly the same scale factor dependence. The familiar dark radiation term of Z_2 symmetric brane worlds is also found in the MC approach—it is the term $2G_5M/a^4$—and the two approaches are seen to lead to the same "dark" fluids on the brane.

Next one can compare the coefficients of the various terms in Eq. (4.19) and (4.17). How are the four parameters $(q, \tilde{E}, M, \Lambda)$ parametrizing the geodesic motion of the test brane in Sch-AdS$_5$ related to the five parameters $(e_4, M_\pm, \Lambda_\pm)$ in the JC approach? Clearly this identification will force two of these last six parameters to be related. However, before making this identification note one final important point: it is the application of junction conditions which gives rise to the term proportional to G_5^2 in (4.18), and we should not expect such a term in the MC approach. Comparing (4.17), (4.18), and (4.19) this is indeed verified. Furthermore, since both q and e_4 are brane charges, we expect $q \propto e_4$ so that one deduces that

$$\Lambda = \langle\Lambda\rangle, \qquad M = \langle M\rangle. \tag{4.20}$$

[9] The brane tension λ is related to the parameter T_∞ of Carter and Uzan (2001) by $T_\infty = 4\lambda/3\pi$.

Thus the mass M, for example, of the Schwarzschild black hole in must be identified with the average mass $\langle M \rangle$ in the JC approach. Then, since L is independent of M, it ought to be independent of $[M]$, and this forces $L^4 \propto \lambda^{-1}$. Here the constant of proportionality is arbitrary because of the freedom in how E is defined and because q is also scaled by the multiplicative constant relating $A_{\mu\nu\rho\sigma}$ and $C_{\mu\nu\rho\sigma}$. Thus we are free to write

$$\frac{\tilde{E}}{L} = [M] \qquad \frac{2q}{L^5} = \pi^2 e_4 \langle F \rangle \tag{4.21}$$

which forces

$$\langle \Lambda \rangle = -4\sqrt{6\pi\lambda}. \tag{4.22}$$

4.3. Effects of Angular Momentum

We now return to the full 10D case of section 4.1 and consider nonzero angular momentum, $\ell \neq 0$. Now the Friedmann equation (4.14) has two extra contributions. The first is proportional to a^{-6} and is characteristic of an equation of state $w \equiv p/\rho = 1$. However it contributes with a negative energy density in (4.14). The second is proportional to a^{-10} which would correspond to matter with equation of state $w = 7/3$. To understand the effect of these terms, it is helpful to construct an effective potential for the brane motion as a function of the radial coordinate r.

Our approach is the same as that used when considering planetary orbits: the constants of the motion are used to eliminate all but the radial degree of freedom. Then the 1D equation of motion follows from the Lagrangian, $L = \frac{1}{2}\dot{r}^2 - V_{\mathrm{eff}}(r)$, where $V_{\mathrm{eff}} \equiv E - \frac{1}{2}\dot{r}^2$.

Here, two effective potentials can be constructed. The first, V^t_{eff}, determines the brane dynamics as seen by an observer outside the brane whose time coordinate is t:

$$\begin{aligned} V^t_{\mathrm{eff}}(r, \ell, E) \equiv E - \frac{1}{2}\dot{r}^2 \;&=\; E + \frac{\mathcal{A}}{2\mathcal{B}}\left[1 + \frac{\mathcal{A}}{\mathcal{B}}\frac{(\ell^2 - \mathcal{C})}{(E + \mathcal{E})^2}\right] \\ &= E - \frac{1}{2}\left(\frac{r}{L}\right)^4 f^2\left[1 - r^2 f \frac{(\ell^2 L^4 + r^6)}{(\tilde{E}L^4 + qr^4)^2}\right], \end{aligned} \tag{4.23}$$

where we have used (4.9) followed by (4.12). In a similar way, the second potential V^τ_{eff}, which is defined for an observer living on the brane, is given by

$$V^\tau_{\mathrm{eff}}(r, \ell, E) \equiv E - \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = E + \frac{g_d^3}{2\mathcal{A}\mathcal{B}\mathcal{C}}\left[\mathcal{C}(E + \mathcal{E})^2 + \mathcal{A}(\ell^2 - \mathcal{C})\right]$$

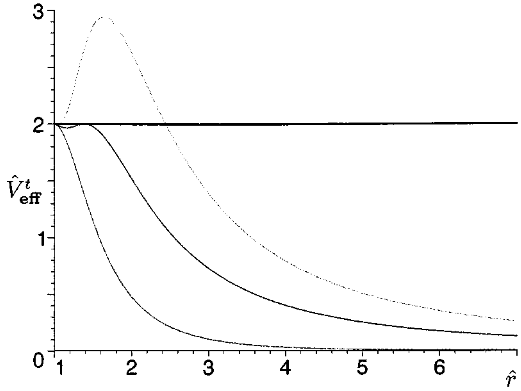

Fig. 1. The rescaled effective potential $\hat{V}^t_{\text{eff}}$ for $q = 1$. The parameters are $\hat{E} = 2$ and $\hat{L} = 1$. The lower curve has $\hat{\ell} = 0$, the upper one $\hat{\ell} = 5$, and the middle one $\hat{\ell} = \hat{\ell}_c = 3.49$.

$$= E + \frac{1}{2}\left(\frac{L}{r}\right)^6 \left[f\frac{r^2}{L^4}\left(\ell^2 + \frac{r^6}{L^4}\right) - \left(\tilde{E} + q\frac{r^4}{L^4}\right)^2\right]. \quad (4.24)$$

This second potential is more relevant for cosmology. Initially, however, we study both potentials, focusing on BPS branes for which $q = 1$ (for $q \neq 1$ see (Boehm and Steer, 2002)).

First consider some properties of V^t_{eff}. Since $f = 0$ at the horizon ($r = r_0$), it follows from (4.23) that $V^t_{\text{eff}}(r_0) = E$ and $\partial V^t_{\text{eff}}/\partial r\ |_{r=r_0} = 0$. Hence the potential has a turning point at the horizon. Also $V^t_{\text{eff}}(r \to \infty) = 0$. Thus only when $E = 0$ does the brane have zero kinetic energy at infinity. The behavior of V^t_{eff} between the horizon and infinity depends on the size of ℓ^2. This is illustrated in Fig. 1 where we have introduced the rescaled quantities: $\hat{V}^t_{\text{eff}} = V^t_{\text{eff}} L^4/r_0^4$, $\hat{E} = EL^4/r_0^4$, $\hat{r} = r/r_0$, $\hat{L} = L/r_0$ and $\hat{\ell} = \ell L^2/r_0^3$.

If $\ell = 0$ (the lower line in the figure) and the brane is moving radially inwards, it reaches the horizon as $t \to \infty$ where it is "absorbed" by the black hole. Alternatively, a brane initially moving radially outwards escapes to infinity.

If the brane has a large angular momentum ℓ, as in the upper curve of Fig. 1, then a centrifugal potential barrier forms. Thus if the brane initially moves inwards from infinity, it bounces back at a given radius to move back out to infinity. On the other hand, the brane could also be trapped in the small region near the horizon (see Fig. 2). Suppose that a brane moves radially outwards in this region: it continues moving outwards until it is reflected off the potential barrier eventually being absorbed by the black hole.

There is a critical value of the angular momentum ℓ_{c} and corresponding critical radius $r_{\text{c}} > r_0$ for which $V^t_{\text{eff}}(r_{\text{c}}) = E$ and $\partial V^t_{\text{eff}}/\partial r\ |_{r=r_{\text{c}}} = 0$. (See the

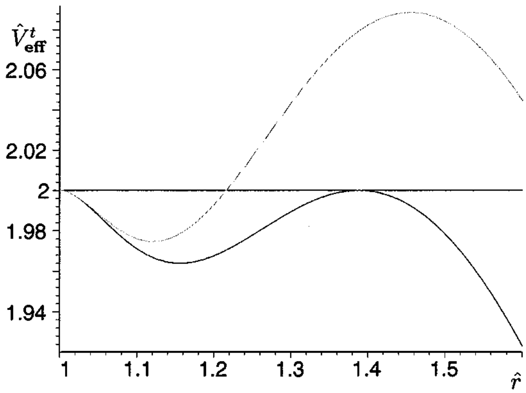

Fig. 2. Detail of the rescaled effective potential $\hat{V}^t_{\text{eff}}$. The parameters $\hat{E}$ and $\hat{L}$ take the same values as in Fig 1. The lower curve has $\hat{\ell} = \hat{\ell}_c = 3.49$ and the upper one $\hat{\ell} = 3.7$. The critical radius is $\hat{r}_c = 1.39$.

middle curve in Fig. 1 and the lower one in Fig. 2). With this angular momentum the brane may reach a stable circular orbit with radius r_c. The expression for ℓ_c is given in the appendix.

Now consider the effective potential V^τ_{eff} of Eq (2.24). This describes the brane trajectory as a function of brane-time τ, and since $r(\tau) = a(\tau)$, also the behavior of the scale factor. Notice that $V^\tau_{\text{eff}}(r_0) = E - \frac{1}{2}X^{-6}(E + \frac{1}{2}X^4)^2 \neq 0$ is ℓ independent, and that as $r \to \infty$, $V^\tau_{\text{eff}} \to E$. Hence in this limit $(dr/d\tau)^2 = 0$, which reflects the fact that there is no cosmological constant in this case (of $q = 1$). Furthermore, observe that the coefficient of the ℓ^2-term is positive and is given by $fL^2/2r^4$; this is responsible for the centrifugal barrier.

The behavior of V^r_{eff} as a function of r is shown in Fig. 3. Again one identifies three regimes:

- $\ell < \ell_c$. The universe either expands or contracts forever. Expansion/contraction depends on whether the brane initially moves radially outwards/ inwards.
- $\ell > \ell_c$. Here there are two possibilities. (i) The universe initially contracts —corresponding to the brane moving in from infinity—before bouncing off the centrifugal barrier and starting a period of expansion. (ii) The brane moves radially outwards from the horizon, expanding at the same time, and then bounces off the centrifugal barrier. It then contracts before it terminating its life after some finite brane time inside the black hole—a "black crunch."

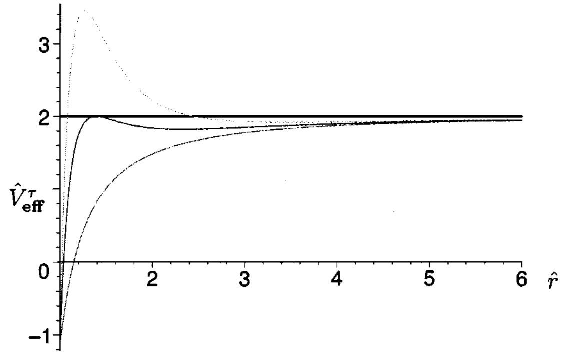

Fig. 3. The rescaled effective potential $\hat{V}^{\tau}_{\text{eff}}$ for the same values of the parameters as in Fig. 1.

- $\ell = \ell_c$. If the brane moves radially inwards from infinity, the universe will contract but the rate of contraction will decrease until, after an infinite amount of time, the scale factor takes the constant value $a = r_c$. If, on the other hand, the brane moves radially outwards from the horizon then it expands but again the scale factor reaches the value $a = r_c$. When curvature is included this can lead to cyclic universes (Brax and Steer, 2002a)

Allied with the question of brane dynamics is the question of brane initial conditions. Clearly for an expanding solution we require the brane to moving outwards from the black hole, but this begs the question of how the brane came to be in this state. One interesting idea is to suppose the brane is Hawking emitted from the black hole. However, the probability of such an event is thought to be extremely low (Maldacena, 1996).

4.4. Radiation on the Brane

So far we have focused on the dynamics of branes with no matter on them. In this subsection we comment very briefly on branes with radiation, that is $F_{ij} \neq 0$ (see (Kehagias and Kiritsis, 1999)). The important point is that the Friedmann equation resulting from the MC approach contains no ρ_{rad}^2 terms, but only terms linear in ρ_{rad} (the energy density in radiation on the brane). In contrast to the back-reaction of the brane on the bulk metric is responsible for the ρ_{rad}^2 terms in the JC approach (Binétruy, *et al.*, 2000a,b; Kraus, 1999).

As mentioned in section 4.1, if electromagnetic fields are present on the brane their nonzero energy density affects the brane dynamics via the action (3.1). Following Kehagias and Kiritsis (1999) we now determine the effect of a uniform electric field $\langle \mathbf{E}^2 \rangle$ on the brane dynamics and hence its contribution to the Friedmann equation.

From (3.1), the Lagrangian is now

$$\mathcal{L} = -\sqrt{\mathcal{A} + \mathcal{B}\dot{r}^2 + \mathcal{C}h_{IJ}\dot{\phi}^I\dot{\phi}^J - \mathbf{E}^2 g_d^2} + \mathcal{E} \equiv -\sqrt{Z} + \mathcal{E} \tag{4.25}$$

where $\mathbf{E}^2 = 2\pi\alpha' E_i E^i$ and $E_0 = -\dot{A}_i$ in the gauge $A_0 = 0$. The equation of motion for E_i gives (Kehagias and Kiritsis, 1999)

$$\mathbf{E}^2 = \frac{\mu^2 Z}{g_d^4}, \tag{4.26}$$

where $\mu^2 = 2\pi\alpha' \mu_i \mu^i$ and μ_i are integration constants. Solving for $\mathbf{E}^2$ yields

$$\mathbf{E}^2 g_d^2 = \left(\frac{\mu^2}{\mu^2 + g_d^2}\right)\left(\mathcal{A} + \mathcal{B}\dot{r}^2 + \mathcal{C}h_{ij}\dot{\phi}^I\dot{\phi}^J\right),$$

so that (4.25) becomes

$$\mathcal{L} = -\sqrt{\mathcal{A}' + \mathcal{B}'\dot{r}^2 + \mathcal{C}'h_{IJ}\dot{\phi}^I\dot{\phi}^J} + \mathcal{E}, \tag{4.27}$$

where $\mathcal{A}' = \mathcal{A}\left(1 + \mu^2 g_d^{-2}\right)^{-1}$ and identical relations hold for $\mathcal{B}'$ and $\mathcal{C}'$. Hence the expressions for $\dot{r}^2$ and $h_{IJ}\dot{\phi}^I\dot{\phi}^J$ are just as in (4.9), but with $\mathcal{A} \to \mathcal{A}'$, etc. Furthermore, it is straightforward to show that $dt^2 = g_d^3 \mathcal{A}^{-2}(E + \mathcal{E})^2(1 + \mu^2 g_d^{-2})d\tau^2$ so that the Friedmann equation including the effects of radiation is

$$H_{\text{total}}^2 = H^2 + H_{\langle \mathbf{E}^2 \rangle}^2, \tag{4.28}$$

where H^2, which is independent of μ^2, is given in Eq. (4.10), and

$$\begin{aligned} H_{\langle \mathbf{E}^2 \rangle}^2 &= -\frac{1}{4\mathcal{A}\mathcal{B}\mathcal{C}} \frac{(g_d')^2}{g_d} \mu^2 \left[\mathcal{C}(E + \mathcal{E})^2 + \mathcal{A}\ell^2\right] \\ &= \rho_{\text{rad}} \left(qL + \frac{\tilde{E}L^5}{a^4}\right)^2 + \ell^2 \rho_{\text{rad}} \left(\frac{L^8}{a^6}\right)\left(\frac{X^4 L^2}{a^4} - \frac{1}{L^2}\right). \end{aligned} \tag{4.29}$$

Here, since $\rho_{\text{rad}} \equiv \mu^2/a^4$ is energy density in radiation, it would appear that the 4D Newton constant should be identified with

$$\frac{8\pi G_4}{3} = q^2 L^2 = \frac{16q^2}{(-\Lambda)}. \tag{4.30}$$

Hence, to summarize, the final result (now setting $q = 1$) is that

$$\begin{aligned} H^2 &= \frac{8\pi G_4}{3}\rho_{\text{rad}} + \Delta \\ \Delta &= \frac{X^4 L^2}{a^4} + (\rho_{\text{rad}} L^4 + 1)\left[\left(\frac{\tilde{E}L}{a^4}\right)\left(2L + \frac{\tilde{E}L^5}{a^4}\right) + \ell^2 \frac{L^4}{a^6}\left(\frac{X^4 L^2}{a^4} - \frac{1}{L^2}\right)\right] \end{aligned} \tag{4.31}$$

As commented above, this is linear in ρ. If Eq. (4.31) were to describe a realistic cosmology then one could now proceed to try to constrain ℓ, $\tilde{E}$, L and r_0 by nucleosynthesis constraints.

5. MIRAGE COSMOLOGY IN A ROTATING BLACK HOLE BULK

An aspect of the Sch-AdS$_5\times$S$_5$ background which simplifies considerations is that the scale factor is just the radial distance of the brane from the black hole. When the brane moves in other backgrounds the expression for $a(\tau)$ is generally more complicated: recall from (2.9) that $a^2(\tau) = g_d(X^A(\tau))$.

As an example of this, we consider MC in a rotating black hole background. This supergravity solution was constructed in Kraus *et al.* (1999), and brane dynamics and thermodynamics in this background were studied in (Cai (1999) and Cai and Soh (1999)). The possibility of a varying speed of light effect in MC was addressed in (Alexander (2001) and Kiritsis (1999b)). In this section we introduce a general formalism for studying MC in the background of a rotating source. We clarify and correct aspects in recent literature. Additionally, we consider the dark fluid terms that arise in this model. We comment more fully on varying speed of light effects in section 6.1.

5.1. Background Metric

The metric for the rotating black hole solution is (Kraus *et al.*, 1999),

$$ds^2 = \frac{1}{\sqrt{f}}\left(-hdt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \sqrt{f}\left[\frac{dr^2}{\tilde{h}} - \frac{4ml\cosh\alpha}{r^4\Delta f}\sin^2\theta dt d\phi \right.$$
$$\left. + r^2(\Delta d\theta^2 + \tilde{\Delta}\sin^2\theta\, d\phi^2 + \cos^2\theta d\Omega_3^2)\right] \tag{5.1}$$

where

$$f = 1 + \frac{2m\sinh^2\alpha}{r^4\Delta} \equiv 1 + \frac{\tilde{R}^4}{r^4\Delta}$$
$$\Delta = 1 + \frac{l^2\cos^2\theta}{r^2}$$
$$\tilde{\Delta} = 1 + \frac{l^2}{r^2} + \frac{2ml^2\sin^2\theta}{r^6\Delta f}$$
$$h = 1 - \frac{2m}{r^4\Delta}$$
$$\tilde{h} = \frac{1}{\Delta}\left(1 + \frac{l^2}{r^2} - \frac{2m}{r^4}\right) \tag{5.2}$$

Note that this solution is not singular unlike the 4D Kerr metric.[10] The total (quantized) D-brane charge of the black hole is R where $R^4 = 2m \sinh\alpha \cosh\alpha$.

The black hole rotates in the ϕ-plane and its angular momentum is determined by l: if $l = 0$ the off-diagonal terms in (5.1) vanish and the metric is that of a D3 black brane whose MC was considered in Kehagias and Kiritsis (1999).[11] The metric describing rotation about more than one axis was also constructed in Kraus *et al.* (1999). Together with ϕ, the coordinates r and θ in (5.1) are the usual coordinates describing a 3-sphere. Notice that the metric coefficients are functions only of r and θ but not ϕ. In a similar way to the Sch-AdS$_5\times$S$_5$ metric considered in the previous section, there is a factorized 3-sphere contribution.

The horizon is the surface given by $\tilde{h}(r) = 0$ so that

$$r^2 = r_0^2 = \frac{1}{2}\left(\sqrt{l^4 + 8m} - l^2\right).$$

A second critical surface, the infinite red-shift hyperplane, satisfies $h(r) = 0$. We have

$$r^2 = r_\infty^2 = \frac{1}{2}\left(\sqrt{l^4 \cos^4\theta + 8m} - l^2 \cos^2\theta\right) \tag{5.3}$$

and that $r_\infty \geq r_0$, with equality holding at the poles ($\theta = 0, \pi$).

It is convenient to introduce the function

$$f_0 = 1 + \frac{R^4}{r^4\Delta}. \tag{5.4}$$

Then the 4-form potential for this background is (Kraus *et al.*, 1999),

$$C_4 = -\left(\frac{1-f_0}{f}\right) - \frac{l\sqrt{2m}\,\tilde{R}^2}{r^4\Delta f}\sin^2\theta\dot{\phi}. \tag{5.5}$$

Finally, note that the brane scale factor is given by $a = f^{-1/4}$. However, as we shall see shortly, we consider trajectories in which $\theta = \pi/2$. Then $\Delta = 1$ so that

$$a = \left(1 + \frac{\tilde{R}^4}{r^4}\right)^{-1/4}. \tag{5.6}$$

Thus a is bounded by $a_{\max} = 1$ (as $r \to \infty$) and $a_{\min} = (1 + \tilde{R}^4/r_0^4)^{-1/4}$ (as $r \to r_0$).

[10] The 4D Kerr metric can be obtained from the 10D metric in the following way. First get rid of six dimensions, namely x^i and Ω_3. Then put $\alpha = 0$ (which is not too surprising since α contains the string parameters). Finally, for dimensional reasons, $m \to mr^3$.

[11] We believe, however, that there is a slight misprint in the Friedmann equation obtained in Kehagias and Kiritsis (1999) in that case: Eq. (5.6) of Kehagias and Kiritsis (1999) should read $(E + \xi(1 - a^4))^2/a^8$ rather than $(E + \xi a^4)^2/a^8$.

5.2. Brane Dynamics With No Matter

In the static gauge, substitution of the metric (5.1) into the action (4.5) gives (Alexander, 2000; Cai, 1999)[12]

$$\mathcal{L} = -\frac{1}{f}\left[\sqrt{h - f\omega^2} - 1 + f_0 + \frac{l\,\sin^2\theta}{\sinh\alpha}(1-f)\dot{\phi}\right] \tag{5.7}$$

where we have set $q = 1$ and

$$\omega^2 = \frac{\dot{r}^2}{\tilde{h}} - \frac{4ml\,\cosh\alpha}{r^4\Delta f}\,\sin^2\theta\dot{\phi} + r^2\left(\Delta\dot{\theta}^2 + \tilde{\Delta}\,\sin^2\theta\dot{\phi}^2 + \cos^2\theta\dot{\Omega}_3^2\right). \tag{5.8}$$

It is straightforward to obtain the equations of motion for θ from (5.7) and to show that $\dot{\theta} = \ddot{\theta} = 0$ if either $\sin\theta = 0$ or $\cos\theta = 0$. Hence the brane will remain in the same θ-plane for $\theta = 0, \pi/2$.

If $\theta = 0$, then the coefficient of the $\dot{\phi}$ terms vanish in (5.7) and (5.8) so that one is left with diagonal metrics of the form discussed in (4.4). Since we want to study the effect of black hole rotation on brane dynamics, we choose $\theta = \pi/2$ so that $r_\infty^2 = \sqrt{2m}$ and the Lagrangian becomes

$$\mathcal{L} = -\sqrt{\mathcal{A} + \mathcal{B}\dot{r}^2 + \mathcal{C}\dot{\phi}^2 + 2\mathcal{D}\dot{\phi}} + \mathcal{E} + \mathcal{G}\dot{\phi} \equiv -\sqrt{Z} + \mathcal{E} + \mathcal{G}\dot{\phi} \tag{5.9}$$

where

$$\mathcal{A} = -g_d^3 g_{00} = \frac{h}{f^2}, \qquad \mathcal{B} = -g_d^3 g_{rr} = -\frac{1}{f\tilde{h}},$$
$$\mathcal{C} = -g_d^3 g_{\phi\phi} = -\frac{r^2}{f}\tilde{\Delta}, \qquad \mathcal{D} = -g_d^3 g_{0\phi} = \frac{l\sqrt{2m}}{r^4 f^2}\frac{R^4}{\tilde{R}^2}, \tag{5.10}$$

and from (5.5)

$$\mathcal{E} = \frac{(1-f_0)}{f} = -\frac{R^4}{r^4 f}$$
$$\mathcal{G} = \frac{l}{\sinh\alpha}\frac{(f-1)}{f} = \frac{l\sqrt{2m}}{r^4 f}\tilde{R}^2. \tag{5.11}$$

Notice that the coefficient of $d\Omega_3^2$ in (5.1) vanishes when $\theta = \pi/2$ so that the brane can have no angular momentum about this 3-sphere; hence this is a different set up from the one considered in the previous section. However, the brane does have a conserved angular momentum about the ϕ direction: $\ell = \partial\mathcal{L}/\partial\dot{\phi}$.

From (5.9) the angular momentum ℓ and energy E are given by

$$\ell = \mathcal{G} - \frac{1}{Z^{1/2}}(\mathcal{C}\dot{\phi} + \mathcal{D}), \qquad E = \frac{1}{Z^{1/2}}(\mathcal{C}\dot{\phi} + \mathcal{A}) - \mathcal{E} \tag{5.12}$$

[12] In fact our Lagrangian differs from theirs by an irrelevant overall constant.

so that

$$\dot{\phi} = -\frac{\mathcal{A}\tilde{\mathcal{G}} + \mathcal{D}\tilde{\mathcal{E}}}{\mathcal{M}} \qquad \dot{r}^2 = \frac{(\mathcal{AC} - \mathcal{D}^2)}{\mathcal{B}} \frac{[-\mathcal{A}\tilde{\mathcal{G}}^2 - \mathcal{C}\tilde{\mathcal{E}}^2 - 2\mathcal{D}\tilde{\mathcal{G}}\tilde{\mathcal{E}} + (\mathcal{AC} - \mathcal{D}^2)]}{\mathcal{M}^2} \tag{5.13}$$

where we have defined

$$\tilde{\mathcal{G}} \equiv -\ell + \mathcal{G}, \quad \tilde{\mathcal{E}} \equiv -E - \mathcal{E}, \quad \mathcal{M} \equiv \mathcal{D}\tilde{\mathcal{G}} + \mathcal{C}\tilde{\mathcal{E}}. \tag{5.14}$$

Notice that on setting $\mathcal{D} = \mathcal{G} = 0$, Eqs. (5.13) reduce to (4.9) as required. Finally, the brane time τ is obtained by substituting (5.13) into (2.8) to give

$$d\tau^2 = \frac{1}{g_d^3}(\mathcal{A} + \mathcal{B}\dot{r}^2 + \mathcal{C}\dot{\phi}^2 + 2\mathcal{D}\dot{\phi})\, dt^2 = \frac{1}{g_d^3 \mathcal{M}^2}(\mathcal{AC} - \mathcal{D}^2)^2\, dt^2. \tag{5.15}$$

We are now in a position to construct the different effective potentials defined in section 4.3. The first, $V^t_{\text{eff}}(r, \ell, E)$, is given by

$$V^t_{\text{eff}}(r, \ell, E) = E - \frac{(\mathcal{AC} - \mathcal{D}^2)}{2\mathcal{B}} \frac{[-\mathcal{A}\tilde{\mathcal{G}}^2 - \mathcal{C}\tilde{\mathcal{E}}^2 - 2\mathcal{D}\tilde{\mathcal{G}}\tilde{\mathcal{E}} + (\mathcal{AC} - \mathcal{D}^2)]}{\mathcal{M}^2}. \tag{5.16}$$

In Cai (1999), an attempt was made to study this potential in the limit that $\dot{\phi} = 0 = \ell$. However, it is clear from (5.13) that this is not a consistent choice: if $\ell = 0$ then $\dot{\phi} = 0$ only for a very specific value of r, namely when $\mathcal{AG} = \mathcal{D}(E + \mathcal{E})$. Instead, one should study (5.15) for arbitrary ℓ.

First notice that at the horizon $r = r_0$, $\mathcal{B}^{-1} = 0$ so that $V^t_{\text{eff}}(r_0) = E$ and $\partial V^t_{\text{eff}}/\partial r\big|_{r_0} = 0$. This is just as for the effective potential discussed in section 4.3. Now, however, as $r \to \infty$,

$$V^t_{\text{eff}}(r) \to E + \frac{1}{2E^2}(1 - E^2) \tag{5.17}$$

so that only for $|E| > 1$ will the brane be able to escape from the rotating black hole: whenever $|E| < 1$ the brane is trapped. The behavior of V^t_{eff} as a function of ℓ—the brane angular momentum—is similar to that discussed in section 4.3. For all values of l and m, there is a critical value of the angular momentum ℓ_c above which a repulsive centrifugal barrier forms. At $\ell = \ell_c$ (where ℓ_c is of course a function of l, m and the other parameters) there is a corresponding critical radius $r = r_c$ in which the brane is in a stable circular brane orbit with constant angular velocity $\dot{\phi}_c$. (For any other value of ℓ, the radius r is not constant and hence, from (5.13), $\dot{\phi}$ is not constant either.)

Given the equations of motion above, one may ask if there is a solution in which the the relative position of the brane to the rotating source is constant. In other words, is there a solution $\dot{\phi}_c = \Omega$ where Ω is the angular velocity of the black hole given by $\Omega = \sqrt{2}R^{-2}m^{-1/2}lr_0^2$ (Cai and Soh, 1999; Kraus *et al.*, 1999)? In Cai

(1999) it was assumed that such a solution exists and the thermodynamics of the D3-brane was then studied as a function of r (since for a static probe, its distance to the source can be regarded as a mass scale in the SYM theory (Tseyltin and Yankielowicz, 1999)). In particular, by calculating the entropy and heat capacity of the brane for $\theta = \pi/2$ and $\dot{\phi}_c = \Omega$, it was shown that there are two critical points for which these thermodynamic quantities diverge leading to interesting conclusions regarding the mass scale of scalar fields in SYM theory (Cai, 1999). Our analysis of (5.17) suggests, however, that generically $\dot{\phi}_c \neq \Omega$. For a given set of (l, m, α), the value of $\dot{\phi}_c$ depends on E and there is only one specific value of $E \equiv E_c$ for which $\dot{\phi}_c = \Omega$. If for some reason these specific values of (ℓ_c, E_c) are chosen (this is a set of measure zero) then the radial distance of the brane, $r = r_c$, is also fixed. Hence it does not appear consistent with Eqs. (5.13) to study probe brane thermodynamics by setting $\dot{\phi}_c = \Omega$ and then letting r vary.

We now turn to the cosmologically relevant effective potential $V^{\tau}_{\rm eff}(r, \ell, E)$ given by

$$V^{\tau}_{\rm eff}(r, \ell, E) = E - \frac{1}{2}\frac{g_d^3}{\mathcal{B}(\mathcal{CA} - \mathcal{D}^2)}\left[-\mathcal{A}\tilde{\mathcal{G}}^2 - \mathcal{C}\tilde{\mathcal{E}}^2 - 2\mathcal{D}\tilde{\mathcal{G}}\tilde{\mathcal{E}} + (\mathcal{CA} - \mathcal{D}^2)\right]. \tag{5.18}$$

In the limit $r \to \infty$,

$$V^{\tau}_{\rm eff} \to E - \frac{1}{2}(E^2 - 1) \tag{5.19}$$

so that once again for $|E| < 1$ the brane cannot escape from the rotating black hole. Notice that there is a significant difference between $V^{\tau}_{\rm eff}$ in this rotating black hole bulk and that obtained for the Sch-AdS$_5\times$S$_5$ bulk: there the brane (with $q = 1$) always had zero kinetic energy at infinity since $V^{\tau}_{\rm eff}(r \to \infty) = E$. In other words the cosmological constant on the brane vanished. Here, on the other hand, it is clear from (5.19) that even when $q = 1$, the cosmological constant only vanishes when $E = 1$ (as in that case the brane has no kinetic energy at infinity). This is the first indication that the dark fluid terms in the Friedmann equation will be rather different in this rotating black hole background (see section 5.3).

In the limit $r \to r_0$, $V^{\tau}_{\rm eff} < E$, and notice also that the coefficient of the ℓ^2 term is positive so that once again we expect a centrifugal potential barrier. The brane motion can be summarized as follows

- $E < 1$. Independently of ℓ, the brane will be trapped in a region near the horizon and eventually be absorbed by the black hole.
- $E \geq 1$. Here one has similar behavior to the one in the Sch-AdS$_5\times$S$_5$ background. That is, there is again a critical angular momentum ℓ_c which divides the brane trajectories into two different categories as discussed in section 4.1.

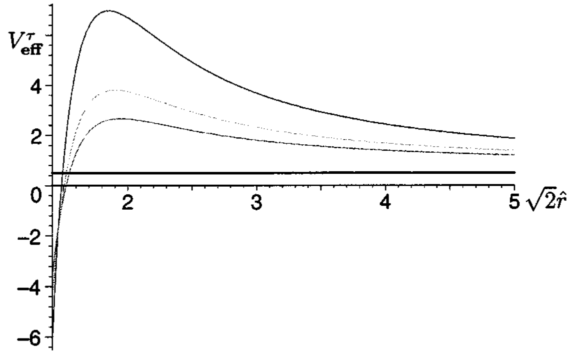

Fig. 4. The effective potential V^{τ}_{eff}. The parameters are $q = 1$, $\hat{l}^2 = 0.5$, $\hat{R}^4 = 4$ (so that $r_0 = 0.88\,r_\infty$), the angular momentum $\hat{\ell}^2 = 0$ and the energy $E = 0.5$. As discussed in the text, the brane will inevitably be trapped and fall into the black hole.

The behaviors are illustrated in Fig. 4 for $E < 1$ and Fig. 5 for $E > 1$, where we have introduced the dimensionless quantities $\hat{r} = r/r_\infty$, $\hat{l} = l/r_\infty$, $\hat{R} = R/r_\infty$ and $\hat{\ell} = \ell/r_\infty$.

In this rotating black hole background, the existence of stable circular orbit with $r > r_0$ was required by Alexander (2000) who studied the effects of a varying speed of light in MC (see section 6.1). In Alexander (2000), however, a rather involved mechanism was constructed to stabilize the brane in a circular orbit at some $r > r_0$ (this was required since the only stable circular orbit was thought to have

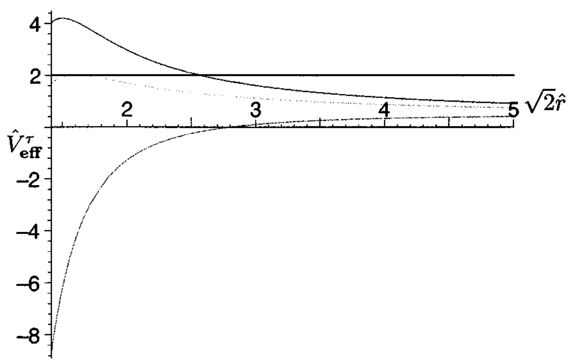

Fig. 5. The effective potential V^{τ}_{eff}. The parameters are $q = 1$, $\hat{l}^2 = 0.5$, $\hat{R}^4 = 4$, and the energy $E = 2$. Once again there are the three possible regimes, depending on the angular momentum. Here the values are $\hat{\ell}^2 = 0$ for the lower curve, $\hat{\ell}^2 = \hat{\ell}^2_c = 2.038$ for the critical curve, and $\hat{\ell}^2 = 2.5$ for the upper curve.

radius $r = r_0$ where the speed of light vanishes (see section 6.1)). According to our analysis, a stable circular orbit with $r > r_0$ always exists when $\ell = \ell_c$—and given our analysis of section 4 this is indeed the case whether or not the black hole rotates.

5.3. Friedmann Equation

We end this section with a few comments regarding the dark fluid terms which appear in the Friedmann equation when doing MC in this rotating black hole background. As mentioned above, we expect an E-dependent cosmological constant. Combination of Eqs. (5.15),(2.10), and (5.13) yield the following Friedmann equation:

$$H^2 = \frac{g_d (g_d')^2}{4\mathcal{B}(\mathcal{C}\mathcal{A} - \mathcal{D}^2)} \left[-\mathcal{A}\tilde{\mathcal{G}}^2 - \mathcal{C}\tilde{\mathcal{E}}^2 - 2\mathcal{D}\tilde{\mathcal{G}}\tilde{\mathcal{E}} + (\mathcal{C}\mathcal{A} - \mathcal{D}^2) \right]. \tag{5.20}$$

Unfortunately, the right-hand side of this equation cannot simply be written as a sum of terms of the form $1/a^p$ for some power p, since now r is not a simple function of the scale factor a: inversion of (5.6) yields

$$r = \tilde{R} \frac{a}{(1 - a^4)^{1/4}}. \tag{5.21}$$

Thus it is not possible simply to read off the dark fluid terms; we conclude that these terms are background dependent (the same conclusion was reached in Kehagias and Kiritsis (1999) as a result of studying a number of different static backgrounds). However, one can study the behavior of H^2 when $a \ll 1$. Then $r \simeq \tilde{R}a$, and a straightforward Taylor expansion yields

$$H^2 \simeq \frac{c_0 + c_2 a^2 + c_4 a^4 + c_6 a^6}{\tilde{R}^6 a^{10}} \tag{5.22}$$

where

$$\begin{aligned} c_6 &= \tilde{R}^6 (E^2 - 1) \\ c_4 &= \tilde{r}^4 \left(l^2 E^2 - \ell^2 - \frac{l^2}{\tilde{R}^2} \right) \\ c_2 &= \tilde{R}^4 (\xi + E)^2 \\ c_0 &= \tilde{R}^2 \left(l^2 (1 + E\xi)^2 + \ell^2 (\xi^2 - 1) \right) - 2 l \ell \sqrt{2m} (1 + \xi E) \end{aligned}$$

and

$$\xi = \frac{R^4}{\tilde{R}^4} = \frac{\cosh\alpha}{\sinh\alpha} = \sqrt{1 + \frac{2m}{\tilde{R}^4}}.$$

Hence as in the case of the Sch-$AdS_5 \times S_5$ black hole of section 4, for small a, H^2 contains dark fluid terms proportional to a^{-10}, a^{-8}, a^{-6}, and a^{-4}. However, it is important to notice now that even if the angular momentum ℓ of the brane vanishes, there are still the contributions going as a^{-6} and a^{-10} in the Friedman equation. These are now sourced by the angular momentum ($\propto l$) of the black hole itself rather than that of the brane.

6. COMMENTS ON CAUSALITY AND BRANE INFLATION

6.1. Causality

In brane world scenarios it is well known that Lorentz invariance is violated (Caldwell and Langlois, 2001). This reflects the fact that gravitons can propagate in the bulk whereas photons are confined to the brane: hence gravitational and light signals generally take different times to propagate between two given points on the brane. In the context of MC, varying speed of light effects have been discussed by Kiritsis (1999b) and Alexander (2002) for slowly moving branes (we will be more specific about the meaning of "slowly moving" below). Our aim here is therefore to comment briefly on this varying speed of light, c_{eff}, without making any approximation regarding the brane dynamics since this was obtained exactly in sections 4.1 and 5.2. (In fact it is not entirely clear to us whether this "varying speed of light" effect should not referred to as a redshift effect. However we use the terminology "varying speed of light" as in Alexander (2000) and Kiritsis (1999b).

In Albrecht and Magueijo (1999), an investigation was made of the cosmological problems which may be resolved if c_{eff} was always larger in the past. Thus in MC we search for an expanding universe for which c_{eff} is always a decreasing function stabilizing at a constant value c_0 as $\tau \to \infty$. Notice that in Alexander (2000) the universe-brane was always considered to be approaching the black hole (corresponding to a contracting universes). We consider the case when the brane moves radially outwards and hence expands: how does c_{eff} behave in that case?

When photons are present on the brane $F_{ij} \neq 0$ (we still keep $B_{ij} = 0$). Expansion in powers of $\alpha' \ll 1$ of the D-brane action (3.1) in the static gauge for $q = 1$ yields, to second order,

$$S \simeq -\lambda \int d^4x \sqrt{-\det \gamma_{ij}} - \lambda \int d^4x \hat{C}_4 \tag{6.1}$$

$$+(2\pi\alpha')^2 \frac{\lambda}{4} \int d^4x \sqrt{-\det \gamma_{ij}} \times \text{tr}\left[\gamma^{-1}\mathbf{F}\gamma^{-1}\mathbf{F}\right] + \cdots. \tag{6.2}$$

Here $\mathbf{A} \equiv A_{ij}$, and note that the term linear in α' gives no contribution since $\mathbf{F}$ is antisymmetric so that $\text{tr}\gamma^{-1}\mathbf{F} = 0$.

Since $\alpha' \ll 1$, to first order the brane dynamics will be governed by the terms in line (6.1) and hence will be given by (4.9) for a Sch-AdS$_5 \times$S$_5$ background, or by (5.13) for the rotating black hole background. Furthermore

$$\mathrm{tr}\left[\boldsymbol{\gamma}^{-1}\mathbf{F}\boldsymbol{\gamma}^{-1}\mathbf{F}\right] = \gamma_d^{-1}\left[-\gamma^{00}\mathbf{E}^2 + \gamma_d^{-1}\mathbf{B}^2\right] \tag{6.3}$$

where $F_{0i} = E_i$ and $F_{ij} = \epsilon^{ijk} B_k$ and we have defined $\gamma_{ab} \equiv \gamma_d \delta_{ab}$. Hence (6.2) is a kinetic term for the gauge fields:

$$(2\pi\alpha')^2 \frac{\lambda}{4} \int d^4x \sqrt{-\det \gamma_{ij}} \times \mathrm{tr}\left[\boldsymbol{\gamma}^{-1}\mathbf{F}\boldsymbol{\gamma}^{-1}\mathbf{F}\right] = (2\pi\alpha')^2 \frac{\lambda}{2} \times \int d^4x \left(-\tilde{A}\mathbf{E}^2 + \tilde{B}\mathbf{B}^2\right) \tag{6.4}$$

where

$$\tilde{A} = \left(\frac{\gamma_d}{|\gamma_{00}|}\right)^{1/2} \qquad \tilde{B} = \left(\frac{|\gamma_{00}|}{\gamma_d}\right)^{1/2}, \tag{6.5}$$

so that the effective speed of light is

$$c_{\mathrm{eff}} = \left(\frac{|\gamma_{00}|}{\gamma_d}\right)^{1/2} \tag{6.6}$$

Here we have used the definition of $\mathcal{L}$ given in (4.6). Notice that $\mathcal{L}$ and C_4 must be evaluated on the brane trajectory $X(\tau)$ so that the effective speed of light clearly depends on the dynamics of the brane itself.

Since we search for a scenario in which c_{eff} tends asymptotically to c_0 as the universe expands, there are two cases to consider: (i) either $\ell < \ell_c$ so that the universe expands reaching $r \to \infty$ as $\tau \to \infty$, or (ii) $\ell = \ell_c$ in which case the scale factor stabilizes at a finite value of $r = r_c$. We analyze these cases first for the nonrotating Sch-AdS$_5 \times$S$_5$ background of section 4.

6.1.1. *Varying Speed of Light in Sch-AdS$_5 \times S_5$*

In this case Eq. (6.6), becomes

$$c_{\mathrm{eff}} = \frac{1}{g_d^2}\sqrt{\left(\mathcal{A} + \mathcal{B}\dot{r}^2 + \mathcal{C}h_{IJ}\dot{\phi}^I\dot{\phi}^J\right)}. \tag{6.7}$$

If the brane moves slowly (Alexander, 2000), then the first term in the square-root dominates and $c_{\mathrm{eff}} \simeq \sqrt{\mathcal{A}}/g_d^2 = \sqrt{f(r)} = \sqrt{1 - (r_0/r)^4}$ (where we have used (4.1) and (4.7)). This result is independent of E and ℓ (respectively the energy and angular momentum of the brane) and of L. Also $c_{\mathrm{eff}} = 0$ at $r = r_0$ and $c_{\mathrm{eff}} \to 1$ as $r \to \infty$ so that c_{eff} increases as the universe expands.

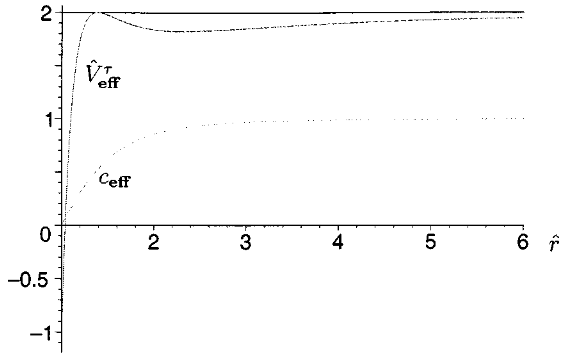

Fig. 6. The rescaled effective potential V^{τ}_{eff} for $\hat{\ell} = \hat{\ell}_c$ (upper curve) and effective speed of light c_{eff} (lower curve) in a Sch-$AdS_5 \times S_5$ bulk. The parameters are as in Fig. 3 so that $q = 1$, $\hat{L} = 1$, and $\hat{E} = 2$. The upper horizontal line is the energy $\hat{E} = 2$.

If the brane does not move slowly, substitution into (6.7), of the specific expressions for $\dot{r}^2$ and $h_{IJ}\dot{\phi}^I\dot{\phi}^J$ given in (4.9) yield

$$c_{\text{eff}} = \frac{1}{g_d^2}\frac{\mathcal{A}}{E+\mathcal{E}} = \left[1 - \left(\frac{r_0}{r}\right)^4\right]\left(\frac{\tilde{E}L^4}{r^4} + 1\right)^{-1}. \tag{6.8}$$

Hence c_{eff} now depends on the energy E of the brane, though not on its angular momentum ℓ. However, we still have that $c_{\text{eff}} = 0$ at $r = r_0$ and that $c_{\text{eff}} \to 1$ as $r \to \infty$. Also, from (6.8), c_{eff} is a strictly increasing function of r. Figure 6 shows c_{eff} and the effective potential V^{τ}_{eff} when $\ell = \ell_c$. Thus once again, as the brane moves outwards from the horizon and expands, c_{eff} decreases. Furthermore, if $\ell = \ell_c$ then c_{eff} stabilises at a value near $1/2$. Hence the only way in which c_{eff} can decrease and stabilize at late times is if $\ell = \ell_c$ and the universe *contracts*.

6.1.2. *Varying Speed of Light in the Rotating Black Hole Background*

A similar analysis for the bulk of section 5 yields

$$c_{\text{eff}} = \frac{1}{g_d^2}\frac{\mathcal{A}\mathcal{C} - \mathcal{D}^2}{\mathcal{M}}.$$

Here $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$ are given in (5.10), $\mathcal{M}$ in (5.14) and we have chosen $\theta = \pi/2$ as in section 5.2. Notice that c_{eff} is now a function of both E and ℓ.

In the limit $r \to \infty$, $c_{\text{eff}} \to 1/E$ (this reflects the E-dependent cosmological constant in this case). One can also show that c_{eff} increases as r increases. Hence c_{eff} tends asymptotically to 1 as $r \to \infty$ only if $E = 1$. For this rotating black hole background we do not present plots of c_{eff} corresponding to the different curves in Fig. 5; the overall behavior of $c_{\rm eff}$ is similar to that shown in Fig. 6. For the parameters of Fig. 5 with $\ell = \ell_{\rm c}$, $c_{\rm eff} \sim 0.2$ at $r = r_{\rm c}$ and then tends asymptotically to $1/2$ as $r \to \infty$. Thus once again, the only way in which $c_{\rm eff}$ can decrease and stabilize at late times is if $E \geq 1$, $\ell = \ell_{\rm c}$ and the universe contracts.

6.2. Comments on Nonrelativistic Brane Matter and Inflation

So far only the effects of radiation on the brane have been considered in MC. (To the best of our knowledge this is also true of the rest of the literature on MC.) How can nonrelativistic matter be included starting from the D-brane action? The answer probably lies in the fermionic sector of the string action which has not been considered here.

As an alternative, one can take a more phenomenological approach and add by hand matter on the brane with an arbitrary equation of state (Parry and Steer, 2001). A byproduct of this approach is that, depending on the bulk, it is possible to show that inflation can occur on the brane (i.e. $a \sim \tau^{\alpha}$ with $\alpha > 1$) (Parry and Steer, 2002), but not in the bulk. An interesting realization of this occurs in the following case: suppose the bulk is generated by a brane gas (Alexander *et al.*, 1999), and consider the late time behavior of this gas. The bulk metric, which is assumed to be flat and roughly homogeneous and isotropic, is described by a scale factor $a(t)$ and depends on the dilaton field $\phi(t)$ and bulk matter parameters ρ and p (energy density and pressure respectively). With the standard embedding, a 3-brane moving in the bulk sees an induced scale factor $a(t(\tau))$, where τ is the brane time. It is the difference between τ and t which is responsible for the different evolutions of the brane and the bulk.

In this scenario, one considers a phenomenological brane action of the form

$$S = \int d^4x \sqrt{-\gamma}\mathcal{L} = \int d^4x \sqrt{-\gamma}\left\{e^{-\phi}\lambda + \xi e^{-m\phi}\mathcal{L}_b\right\}$$

rather than (3.1), where m and ξ are dimensionless constants which determine the coupling of the dilaton to the brane matter $\mathcal{L}_b$. Notice that any coupling to a 4-form has been neglected. The first term above is just the general expression for the kinetic term of (3.1) for nonzero dilaton. Given this action it is not hard to solve for the brane dynamics, and hence to obtain the brane scale factor in the way outlined in section 2. Furthermore, if the brane initially has a *large* velocity (e.g., it is formed as the result of a collision process—say a $\bar{5} - 5$ brane annihilation (Alexader, 2001; Majumder and Sen, 2000)) and if $\rho_b \gg \lambda$ then inflation may occur on the brane in the radiation dominated epoch: in Parry and Steer (2002) this setup is analyzed in detail. The result is that for the natural coupling to the dilaton, $m = 1$, the brane inflates when the bulk is comprised of stiff matter.

A rather different realization of inflation in MC comes from observing that if the brane moves slowly then the action (3.1) may be expanded in powers of $\dot{r}^2$ leading, to first order, to action quadratic in $\dot{r}^2$. The field r can then be identified with the inflaton which now has an unusual kinetic term, and combined with the potential term it can lead to inflation. Indeed the resulting setup is reminiscent of that of Burgess *et al.* (2001), and this approach can be used to study brane inflation in a bulk for which supersymmetry is broken, and inflation is ended by tachyon condensation (Brax and Steer, 2002b).

7. CONCLUSIONS

In this work we have tried to summarize some aspects of the approach to brane cosmology known as mirage cosmology (Kehagias and Kiritsis, 1999). Here the brane is a D3-brane in type IIB string theory and it moves in a 10D bulk metric. As opposed to the 5D junction condition approach to brane cosmology, the D3-brane is treated as a test brane and hence it is straightforward to consider more than one extra dimension.

As explained in section 2, brane motion can induce an effective cosmology on the brane, and once the dynamics of the brane is determined the corresponding Friedmann equation can be obtained. Those parts of the Friedmann equation solely generated by the motion of the brane (and not by matter on the brane) are the dark fluid terms. We have tried to see how the familiar dark radiation term (Binétruy *et al.*, 2000a,b) generalizes when there is more than one extra dimension, and this was done for the two specific 10D bulk metrics of sections 4 and 5.

In section 4 we studied the dynamics of the probe D3-brane in a Sch-AdS$_5$ $\times$ S$_5$ bulk for which the brane geodesics are parametrized by a conserved energy E and an angular momentum ℓ about the S_5. For all ℓ we saw that the cosmological constant on the brane vanished if $q \equiv e/\lambda = \pm 1$ corresponding to BPS (anti-)branes. Also the Friedmann equation was found to contain dark fluid terms proportional to a^{-4}, a^{-8} and to $\ell^2 a^{-6}$, $\ell^2 a^{-10}$.

When $\ell = 0$, the brane motion is constrained to Sch-AdS$_5$. There, however, the exact brane dynamics (including the back-reaction of the brane on the bulk) can be calculated (Carter and Uzan, 2001; Ida, 2000; Kraus, 1999). As discussed in section 4.2, the MC results must be compared to a JC calculation in which Z_2 symmetry is broken since the D3-branes couple to the bulk RR field. We saw that the MC and JC Friedmann equations had the same dark fluid terms, and a further analysis of those equations linked the parameters of the MC approach (E etc) to those of the JC approach (Eqs. (4.20)–(4.22)). For that analysis it was important to allow the D3-branes to have an arbitrary RR charge q.

Nonszero angular momentum, $\ell \neq 0$, generated dark fluid terms $\propto a^{-10}$, a^6. As a result different types of brane trajectories were seen to exist depending on whether or not ℓ was greater or smaller than ℓ_c (Fig. 3):

- $\ell < \ell_c$. The brane contracts/expands (corresponding to inward/outward radial motion) for all τ.
- $\ell = \ell_c$. A (contracting) brane moving radially inwards from infinity reaches, after an infinite τ, a critical radius in which it rotates around the black hole in a stable orbit. An expanding brane moving radially outwards from the horizon reaches the same stable critical radius.
- $\ell > \ell_c$. A centrifugal barrier develops. A (contracting) brane moving radially inwards from infinity bounces off this barrier after a finite τ. Then it moves radially outwards and starts to expand. Similarly a (expanding) brane moving radially outwards from the horizon is also reflected by the barrier after a finite τ; it starts moving radially inwards and contracts until it is swallowed by the black hole.

In section 4.4 we commented on the addition of radiation to the brane. The resulting Friedmann equation only contains terms proportional to $\rho_{\rm rad}$ and not $\rho_{\rm rad}^2$ because of the "passive" nature of the MC approach. In principle, given this Friedmann equation, one could try to constrain the different parameters (such as ℓ, E) via nucleosynthesis constraints. One reason for not doing this is the (current) lack of a treatment for nonrelativistic brane matter in this MC approach. Since D3-branes are BPS states, perhaps this problem will be related to providing a prescription for supersymmetry-breaking. How to do all this is an important question for future work.

The purpose of section 5 was to consider a slightly more complicated bulk and to try to see how many of the results presented in section 4 are in fact bulk dependent. We considered the dynamics of a brane in the rotating black hole metric of Eq. (5.1) and found that the main differences with the Sch-$AdS_5 \times S_5$ bulk are

- An E-dependent cosmological constant on the brane which does not vanish when $q = \pm 1$ unless $E = 1$.
- Brane trajectories which were always trapped by the black hole if $E < 1$. For $E > 1$ the three different classes of ℓ-dependent trajectories outlined above were found.
- For $a \ll 1$, dark fluid terms $\propto a^{-10}$, a^{-6}, a^{-8}, and a^{-4}, the first two of which did *not* vanish when $\ell = 0$ since in this case they were sourced by the angular momentum of the black hole itself (i.e. $\propto l$).

Finally for both bulks, we considered the behavior of $c_{\rm eff}$ as the universe expands. In the Sch-$AdS_5 \times S_5$ bulk $c_{\rm eff}$ is ℓ independent (but E dependent) and vanishes at the horizon r_0. As $r \to \infty$, $c_{\rm eff}$ tends to 1. Hence for a brane with $\ell < \ell_c$ (which expands for all τ), the speed of light always increases tending to 1. For $\ell = \ell_c$ the asymptotic value is $c_0 < 1$. Similar behavior holds in the rotating black hole bulk.

There are many interesting aspects of mirage cosmology which we have not studied here. One of these is the question of the initial singularity (Kehagias and Kiritsis, 1999), and another is an interpretation of the results presented here (and especially the role of the critical angular momentum ℓ_c) in the context of SYM theory and black hole thermodynamics.

A final crucial ingredient, which is required for MC, is a description of brane self-gravity (see Brax and Steer 2002b). As was discussed in the introduction, our approach is to treat brane motion as similar to planetary motion. In the latter case, one usually leaves the question of self-gravity to the geophysicists. However, as cosmologists, it is necessary to know about the internal evolution of the brane. Since it is extremely unlikely this will be induced solely by the motion of the brane in its background, we require an understanding of how local energy density on the brane sources gravity on the brane. Before mirage cosmology can become a fully-fledged cosmology, this vital question must be addressed.

8. APPENDIX

In section 4.3 we commented that there is a critical value of the brane angular momentum, ℓ_c, for which the effective potential has a local maximum at $V_{\text{eff}} = E$. A brane with this angular momentum can be in a circular orbit about the black hole at $r = r_c$.

Mathematically, this situation arises when $V_{\text{eff}} = E$ has repeated roots. If we use the rescaled quantities and let $x = \hat{r}^2$, then from (4.23) or (4.24) this is equivalent to considering when the polynomial $p(x) = (x^2 - 1)(\hat{\ell}^2 + x^3) - x(\hat{E} - \frac{1}{2} + x^2)^2 = 0$. Simplifying, it is left to show that there are repeated roots of the cubic equation

$$yx^3 - \hat{\ell}^2 x^2 + \frac{1}{4}(y-1)^2 x + \hat{\ell}^2 = 0,$$

where $y = 2\hat{E}$. This occurs when

$$\hat{\ell}_c^4 = \frac{1}{128}\left[-y^4 + 76y^3 + 282y^2 + 76y - 1 + \sqrt{(y+1)^2(y^2+34y+1)^3}\right].$$

The expression for the repeated root $x_c = \hat{r}_c^2$ is too complicated to write down here however.

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